# APPROXIMATE ANALYTIC RELATIONSHIP FOR THE EFFICIENCY OF A THIN RADIATING BAR

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An approximate relationship between the efficiency of a thin radiating bar and the dimensionless thermal-conductivity parameter is proposed. The temperature distribution over the bar length is obtained for a wide range of values of this parameter.

We shall consider the efficiency and the temperature distribution for thin bars (fins) of constant cross section and finite length, radiating into a medium at absolute zero temperature; there are no internal heat sources and no incident radiation. A constant temperature $T_0$ is maintained at one end of the bar, while the temperature over a bar cross section is also taken to be constant. We neglect heat exchange at the second end.

The bar efficiency is defined as the ratio of the heat actually emitted to the heat that the bar would supply if its entire surface were at the same temperature, equal to the temperature $T_0$ of the base:

$$\eta = \frac{Q_0}{\sigma \varepsilon u T_0^3 l}. \tag{1}$$

Many studies, including [1, 2] have given graphical relationships for determining the efficiency; here we propose an approximate analytic relationship.

For the given bar, the equation of the steady-state heat-conduction process has the form

$$\frac{d^2T}{dx^2} = \frac{\sigma \varepsilon u}{\lambda f} T^4. \tag{2}$$

Introducing the dimensionless temperature $\theta = T/T_0$ and the length $X = x/l$, we can write (2) as

$$\frac{d^2\theta}{dX^2} = N\theta^4, \tag{3}$$

where

$$N = \frac{\sigma \varepsilon u T_0^3 l^2}{\lambda f} \tag{4}$$

is the bar thermal-conductivity parameter; for a rectangular fin, $N = \sigma\varepsilon T_0^3 l^2/\lambda\delta$.

The heat emitted by the bar is found as

$$Q_0 = -\lambda f \frac{T_0}{l} \left.\frac{d\theta}{dX}\right|_{X=0}. \tag{5}$$

Integrating, we find

$$\frac{d\theta}{dX} = -\sqrt{0.4N(\theta^5 - \theta_l^5)}, \tag{6}$$

where $\theta_l = \theta|_{X=1}$. Then from (1), (5), (6), and (4) we have

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TABLE 1. Bar Efficiency and Temperature at End for Various Values of Parameter N

| $N$ | Found by integration | | From Eq. (8) | | From Eq. (18) | | From Eq. (19) | | From Eq. (20) | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\theta_l$ | $\eta$ | $\eta$ | $\delta\eta$, % | $\theta_l$ | $\delta\theta_l$, % | $\theta_l$ | $\delta\theta_l$, % | $\theta_l$ | $\delta\theta_l$, % |
| 0,03514 | 0,9834 | 0,9565 | 0,9588 | 0,24 | 0,9833 | —0,01 | 0,9834 | 0 | 0,9833 | —0,01 |
| 0,09752 | 0,9578 | 0,8923 | 0,8967 | 0,49 | 0,9573 | —0,05 | 0,9577 | —0,01 | 0,9575 | --0,03 |
| 0,2028 | 0,9224 | 0,8095 | 0,8146 | 0,63 | 0,9212 | —0,13 | 0,9222 | —0,02 | 0,9217 | —0,08 |
| 0,4804 | 0,8561 | 0,6707 | 0,6741 | 0,51 | 0,8540 | —0,25 | 0,8558 | —0,04 | 0,8549 | —0,14 |
| 0,9849 | 0,7807 | 0,5370 | 0,5375 | 0,09 | 0,7801 | —0,08 | 0,7799 | —0,1 | 0,7800 | —0,09 |
| 2,278 | 0,6775 | 0,3880 | 0,3865 | —0,39 | 0,6837 | 0,92 | 0,6717 | —0,86 | 0,6777 | 0,03 |
| 4,017 | 0,6038 | 0,3026 | 0,3010 | —0,53 | 0,6186 | 2,45 | 0,5897 | —2,34 | 0,6041 | 0,05 |
| 4,806 | 0,5806 | 0,2788 | 0,2772 | —0,57 | 0,5985 | 3,08 | 0,5627 | —3,08 | 0,5806 | 0 |
| 6,797 | 0,5362 | 0,2372 | 0,2358 | —0,59 | 0,5610 | 4,62 | 0,5098 | —4,92 | 0,5354 | —0,15 |
| 9,462 | 0,4950 | 0,2025 | 0,2014 | —0,54 | 0,5267 | 6,40 | 0,4588 | —7,31 | 0,4928 | —0,44 |
| 15,14 | 0,4392 | 0,1612 | 0,1604 | —0,50 | 0,4810 | 9,52 | 0,3871 | —11,86 | 0,4341 | —1,16 |
| 20,37 | 0,4058 | 0,1394 | 0,1388 | —0,43 | 0,4539 | 11,85 | 0,3433 | —15,40 | 0,3986 | —1.77 |

$$\eta = \frac{0.6325}{\sqrt{N}} \sqrt{1 - \theta_l^5}. \tag{7}$$

The relationship between the dimensionless temperature $\theta_l$ at the end and the parameter N is found in [1, 2] by numerical integration; graphical relationships are then given for the efficiency as a function of N.

Here we propose an analytic relationship for $\eta = f(N)$, obtained by trial and error. It takes the form

$$\eta = \frac{0.6325}{\sqrt{N + 0.4}} = \frac{1}{\sqrt{2.5N + 1}} \tag{8}$$

and provides sufficient accuracy for engineering purposes, whatever the value of N. Table 1 shows $\eta$ for the N = 0-20 range; the values were found from Eq. (8). For comparison, we have also shown $\eta$ as found from Eq. (7), where $\theta_l$ was found by numerical integration of (6) by the method of [2]; the substitution $y = \sqrt{\theta/\theta_l - 1}$ was made, with integration step $\Delta y = 0.02$. These values are in good agreement with the curves of [1, 2]. Table 1 also shows the values of $\theta_l$.

As we see, the results obtained by means of Eq. (8) differ from those found by integration by no more than 1%. When $N \geq 10$, we can use the simpler formula

$$\eta = \frac{0.6325}{\sqrt{N}}. \tag{9}$$

The proposed relationship (8) can be used to obtain approximate analytic expressions for the heat flux in any bar cross section, and for the temperature distribution over the bar length for a wide range of values of N.

From (1) and (8), the heat flux through the base of the bar is

$$Q_0 = \frac{\sigma\varepsilon u l T_0^4}{\sqrt{2.5N + 1}}. \tag{10}$$

The heat flux through a bar cross section, a distance x from the base is

$$Q_x = \frac{\sigma\varepsilon u l (1 - X) T_0^4 \theta^4}{\sqrt{2.5N (1 - X)^2 \theta^3 + 1}}. \tag{11}$$

On the other hand, with allowance for (6),

$$Q_x = \lambda f \frac{T_0}{l} \sqrt{0.4N (\theta^5 - \theta_l^5)}. \tag{12}$$

Equating the right sides of (11) and (12), after certain algebraic manipulations we obtain

$$\theta^5 - \theta_l^5 = 2.5N (1 - X)^2 \theta^3 \theta_l^5. \tag{13}$$

After substitution of (13) into (6) we have

$$\frac{d\theta}{dX} = -N (1 - X) \theta^{3/2} \theta_l^{5/2}. \tag{14}$$

TABLE 2. Temperature Distribution Over Bar Length for Various Values of Parameter N

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $X$ | 0 | 0,08857 | 0,1797 | 0,2733 | 0,4668 | 0,6670 | 1,0000 |
| $N$=0,4804 | $\theta_1$ | 1,000 | 0,9733 | 0,9493 | 0,9281 | 0,8938 | 0,8706 | 0,8540 |
| | $\theta$ | 0,9994 | 0,9725 | 0,9483 | 0,9270 | 0,8927 | 0,8694 | 0,8549 |
| | $\delta\theta$, % | −0,06 | −0,08 | −0,11 | −0,12 | −0,12 | 0,14 | 0,1 |
| | $X$ | 0 | 0,08901 | 0,2386 | 0,4055 | 0,5884 | 0,7171 | 1,000 |
| $N$=2,278 | $\theta_1$ | 1,000 | 0,9296 | 0,8401 | 0,7702 | 0,7198 | 0,6970 | 0,6775 |
| | $\theta$ | 1,001 | 0,9317 | 0,8419 | 0,7714 | 0,7204 | 0,6974 | 0,6777 |
| | $\delta\theta$, % | 0,10 | 0,23 | 0,21 | 0,16 | 0,08 | 0,06 | 0,03 |
| | $X$ | 0 | 0,06213 | 0,1321 | 0,2509 | 0,4841 | 0,6982 | 1,000 |
| $N$=4,806 | $\theta_1$ | 1,000 | 0,9246 | 0,8570 | 0,7692 | 0,6600 | 0,6062 | 0,5806 |
| | $\theta$ | 0,9929 | 0,9233 | 0,8578 | 0,7708 | 0,6609 | 0,6064 | 0,5806 |
| | $\delta\theta$, % | −0,71 | −0,14 | 0,09 | 0,21 | 0,14 | 0,03 | 0 |
| | $X$ | 0 | 0,08136 | 0,2136 | 0,3313 | 0,4637 | 0,7140 | 1,000 |
| $N$=6,797 | $\theta_1$ | 1,000 | 0,8880 | 0,7628 | 0,6868 | 0,6264 | 0,5599 | 0,5362 |
| | $\theta$ | 0,9796 | 0,8780 | 0,7610 | 0,6860 | 0,6257 | 0,5589 | 0,5354 |
| | $\delta\theta$, % | −2,04 | −1,13 | −0,24 | −0,12 | −0,11 | −0,18 | −0,15 |
| | $X$ | 0 | 0,04449 | 0,1120 | 0,2844 | 0,4018 | 0,7268 | 1,000 |
| $N$=9,462 | $\theta_1$ | 1,000 | 0,9232 | 0,8198 | 0,6792 | 0,6139 | 0,5168 | 0,4950 |
| | $\theta$ | 0,9589 | 0,8963 | 0,8169 | 0,6742 | 0,6102 | 0,5144 | 0,4928 |
| | $\delta\theta$, % | −4,11 | −2,91 | −0,35 | −0,74 | −0,6 | −0,46 | −0,44 |
| | $X$ | 0 | 0,05385 | 0,1190 | 0,2892 | 0,3954 | 0,7415 | 1,000 |
| $N$=15,14 | $\theta_1$ | 1,000 | 0,8872 | 0,7871 | 0,6248 | 0,5626 | 0,4586 | 0,4392 |
| | $\theta$ | 0,9102 | 0,8303 | 0,7518 | 0,6098 | 0,5518 | 0,4526 | 0,4341 |
| | $\delta\theta$, % | −8,98 | −6,41 | −4,48 | −2,40 | −1,92 | −1,31 | −1,16 |
| | $X$ | 0 | 0,02954 | 0,08185 | 0,1690 | 0,3773 | 0,7491 | 1,000 |
| $N$=20,37 | $\theta_1$ | 1,000 | 0,9240 | 0,8198 | 0,6991 | 0,5377 | 0,4237 | 0,405 |
| | $\theta$ | 0,8682 | 0,8220 | 0,7517 | 0,6596 | 0,5206 | 0,4152 | 0,3988 |
| | $\delta\theta$, % | −13,18 | −11,04 | −8,31 | −5,65 | −3,18 | −2,01 | −1,7 |

Note: $\theta_1$ determined by integration, $\theta$ found from Eq. (17).

After separating variables and integrating, we have

$$f(\theta) = \theta^{-1/2} + 0.25N\theta_l^{5/2}(1-X)^2 = C, \quad (15)$$

$$C = f(\theta)|_{X=0} = f(\theta)|_{X=1} = 1 + 0.25N\theta_l^{5/2} = \theta_l^{-1/2}. \quad (16)$$

Then the temperature in any bar cross section x is expressed as

$$\theta = \frac{1}{\left[\theta_l^{-1/2} - 0.25N\theta_l^{5/2}(1-X)^2\right]^2}. \quad (17)$$

The value of $\theta_l$ can be found from (7) and (8):

$$\theta_l = \frac{1}{\sqrt[5]{2.5N+1}}. \quad (18)$$

On the other hand, from (16) and (18) we obtain

$$\theta_l = \frac{1}{\left(1 + \frac{0.25N}{\sqrt{2.5N+1}}\right)^2}. \quad (19)$$

Table 1 shows results obtained from (18) and (19); as we see, the two formulas yield good agreement with the results of numerical integration for the N = 0–2 range (the discrepancy does not exceed 1%). As N increases, the accuracy of the formulas drops; Eq. (18) gives values of $\theta_l$ that are too high, and Eq. (19) gives results that are too low. Over a wide range of N values, we obtain good agreement with the results of integration if we represent $\theta_l$ as the arithmetic mean of the values yielded by Eqs. (18) and (19):

$$\theta_l = 0.5\left[\frac{1}{\sqrt[5]{2.5N+1}} + \frac{1}{\left(1 + \frac{0.25N}{\sqrt{2.5N+1}}\right)^2}\right]. \quad (20)$$

As we see from Table 1, the difference between the values of $\theta_l$ as found from (20) and by integration does not exceed 1% over the N = 0–10 range, and 2% over the N = 10–20 range. This formula can be recommended for determination of $\theta_l$ over the N = 0–20 range.

To find the limits of applicability of Eq. (17) we calculated the actual temperature distribution over the bar length for several values of N. Here we made use of the $\theta_l$—N relationship obtained by numerical integration.

For any cross section x along the bar length, the thermal-conductivity parameter for the bar section $l$ —x can be represented as

$$N_x = N(1-X)^2\theta^3, \tag{21}$$

while the dimensionless temperature of the bar end, represented in terms of the temperature at the cross section x, $\theta_{lX} = T_l/T_X$, will be a function of $N_X$. The temperature at section x is found as

$$\theta = \frac{\theta_l}{\theta_{lx}}. \tag{22}$$

From the value found for $\theta$ and Eq. (20), we can determine the corresponding value of X. Thus specifying a value for $\theta$, for any value of the thermal-conductivity parameter we can successively determine $\theta_{lX}$, $N_X$, X.

Table 2 shows the results of these computations; we have also given the values of $\theta$ found from Eq. (17); the value of $\theta_l$ was found in accordance with (20).

As we see from Table 2, in the 0-5 range of N values, the difference between the results found from (17) and those obtained by integration does not exceed 1%; as N increases further, Eq. (17) gives values of $\theta_l$ that are too low; when N = 10, the difference reaches 5% for small values of X. Results obtained from (17) are also in good agreement with the temperature-distribution curves for fins with N = 0.5 and N = 2 [1]. Formula (17) can be recommended for determination of the temperature distribution over the length of a bar for N = 0-5 throughout the entire range of X, and for bars with N = 5-10 for $X \geq 0.1$.

We can use (14) and (17) to obtain another expression for the heat flux in any bar cross section:

$$Q_x = \frac{\lambda f T_0 N(1-X)\theta_l^{5/2}}{\left[\theta_l^{-1/2} - 0.25N\theta_l^{5/2}(1-X)^2\right]^3}. \tag{23}$$

The limits of applicability of this formula are the same as for (17).

Thus we can use the proposed analytic relationship (8) not only to determine the efficiency of a thin radiating bar for any value of N, but also to derive approximate expressions for the heat-flux and temperature distributions along the bar: (11), (17)-(20), (22), over a fairly wide range of variation in N, encompassing most of the cases found in practice. Equation (8) can also be used to optimize radiating systems.

## NOTATION

| | |
|---|---|
| Q | is the heat flux; |
| T | is the temperature; |
| $\sigma$ | is the Stefan—Boltzmann constant; |
| $\varepsilon$ | is the emissivity; |
| $\lambda$ | is the thermal conductivity; |
| u | is the perimeter; |
| f | is the bar cross section; |
| $2\delta$ | is the thickness of a rectangular fin; |
| $\eta$ | is the efficiency of the bar or fin; |
| x | is the coordinate; |
| $l$ | is the length of the bar or fin; |
| $N = \sigma\varepsilon u T_0^3 l^2/\lambda f$ | is the dimensionless thermal-conductivity parameter; |
| $\theta = T/T_0$ | is the dimensionless temperature; |
| $X = x/l$ | is the dimensionless length. |

### Subscripts

0 refers to parameters at the beginning of the bar;
$l$ refers to parameters at the end of the bar.

## LITERATURE CITED


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